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Some new biautomatic Coxeter groups

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Abstract

A Coxeter system is called *two-dimensional* if every parabolic subgroup of rank at least 3 is infinite. We apply results of Niblo and Reeves and of B.T. Williams to show that Coxeter groups with two-dimensional systems are biautomatic if the corresponding presentation contains no visible Euclidean "triangle" groups. This provides a partial positive answer to a question posed by Niblo and Reeves in [G. Niblo, L. Reeves, Coxeter groups act on CAT(0) cube complexes, J. Group Theory 6 (3) (2003) 399–413].
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1. Introduction

Recall that a *Coxeter system* is a pair (W, S) where W is a group with presentation $\langle S|R\rangle$, for $S = \{s_i\}_{i\in I}$, and

$$R = \left\{(s_i s_j)^{m_{ij}} \;\middle|\; m_{ij} \in \{1, 2, \ldots, \infty\},\ m_{ij} = m_{ji},\ \text{and } m_{ij} = 1 \Leftrightarrow i = j\right\}.$$

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The group W is called a *Coxeter group*, and the elements of S are called the *fundamental generators* corresponding to the system (W, S). We often omit the word 'Coxeter' when discussing Coxeter groups and Coxeter systems.

Given a system (W, S), the group generated by any subset $T \subseteq S$ is called a (*standard*) *parabolic subgroup*, and is denoted W_T. (It is known that W_T is a Coxeter group in its own right, with system (W_T, T). See [4].) The *rank* of the parabolic subgroup W_T is defined to be $|T|$. If W_T is finite, W_T is called a *spherical subgroup* of W (relative to the system (W, S)). The system (W, S) is called *two-dimensional* (abbreviated 2D) if every spherical subgroup has rank at most 2.

We are able to capture all of the information of the presentation $\langle S|R\rangle$ graphically, by means of the Coxeter diagram. The *Coxeter diagram* (or, simply, *diagram*) $\mathcal{V}$ corresponding to (W, S) is an edge-labeled graph whose vertices are in one-to-one correspondence with S, and for which there is an edge $[s_i s_j]$ between s_i and s_j (with label m_{ij}) if and only if $m_{ij} < \infty$.

Let us now address the notion of biautomaticity.

Given a finite set A, denote by A^* the set of (possibly trivial) finite strings (called *words*) of elements of A. A *regular language* over A is a subset $\mathcal{L} \subseteq A^*$ such that $\mathcal{L}$ is accepted by a finite state automaton.

Given two words u and v in A^*, we say that u and v are k-*fellow travelers* if for all t $(0 \leqslant t \leqslant \max\{|u|, |v|\})$, $\mathrm{dist}_A(u(t), v(t)) \leqslant k$. (Here, $\mathrm{dist}_A(x, y)$ is the distance in the word metric relative to A, and $u(t)$ is the prefix of u of length t.)

Now let G be a group with finite generating set A. We say G is *biautomatic* if there exists a regular language $\mathcal{L}$ over A such that

(1) $\mathcal{L}$ surjects onto G (that is, given $g \in G$ there is some word in $\mathcal{L}$ which is equal to g as a group element), and
(2) there is a constant k such that whenever $u, v \in \mathcal{L}$ and $a \in A \cup \{1\}$ such that either $u =_G va$ or $u =_G av$ holds, then u and v are k-fellow travelers.

Remark. This definition is not the standard definition, but they can be shown to be equivalent. (See [7].) Roughly speaking, a group G is biautomatic if we can demonstrate a set of normal forms for the elements of G which can be generated by a finite state automaton, and which are not "too different" from one another for elements of G which are close in the word metric of the group. For nice properties enjoyed by such groups (and by automatic groups), see [7], as well as the splendid survey article by Farb [8].

It is known that all Coxeter groups satisfy the weaker condition of being *automatic* (proven by Brink and Howlett in [6]). It is also known that word-hyperbolic Coxeter groups and right-angled Coxeter groups are biautomatic, and it is reasonable to conjecture that all Coxeter groups are biautomatic.

The goal in this paper is to prove the following fact:

Theorem 1.1. *Let (W, S) be a two-dimensional Coxeter system with diagram $\mathcal{V}$. Suppose that $\mathcal{V}$ has no triangles corresponding to rank-3 affine Euclidean groups (that is, there* *are no triangles in $\mathcal{V}$ with edge label multiset* $\{2,3,6\}$, $\{2,4,4\}$, *or* $\{3,3,3\}$). *Then W is biautomatic.*

Since there are a large number of such groups which are not word-hyperbolic, this result enlarges the collection of Coxeter groups known to be biautomatic.

Remark. The author has recently learned that P.-E. Caprace has proven Theorem 1.1 without the condition that (W,S) be two-dimensional. His proof and the proof provided below were obtained entirely independently of one another and utilize very different techniques.

To prove Theorem 1.1, we modify the method of "centralizer chasing" used in [1–3]. In those papers, the spherical subgroups of W with respect to two different systems are compared. Use of information about the conjugacy of these subgroups, and of the structure of the centralizer $C(s)$ of a fundamental generator s, allowed proofs of various results concerning rigidity (i.e., uniqueness of presentation) and the structure of $\mathrm{Aut}(W)$.

Here again we rely on our knowledge of the structure of $C(s)$, developed in [5]. Conjugacy of W's subgroups will also play a role, as will the following fact, which follows from [9,11]:

Theorem 1.2. *Let (W,S) be a Coxeter system. Then W is biautomatic if for every multiset $\{p,q,r\}$ of positive integers, W contains only finitely many conjugacy classes of subgroups isomorphic to the "triangle group" (p,q,r):*

$$\left\langle a,b,c \;\middle|\; a^2, b^2, c^2, (ab)^p, (bc)^q, (ca)^r \right\rangle.$$

Theorem 1.1 therefore answers the final question posed in [9] in the case of two-dimensional Coxeter groups.

Throughout the remainder of the paper, (W,S) will be a 2D Coxeter system with diagram $\mathcal{V}$. Suppose $s,t \in S$. If $m = m_{st} < \infty$ is even, we define $u_{st} = (st)^{m/2-1}s$. If instead $m = m_{st}$ is odd, we define $v_{st} = (st)^{(m-1)/2}$. Note that u_{st} commutes with t, that $v_{st}^{-1} = v_{ts}$, and that $v_{st} s v_{ts} = t$.

2. Preliminary lemmas

Let $\bar{\mathcal{V}}$ be the graph resulting from $\mathcal{V}$ by removing all edges with even labels. For a fixed vertex $s \in S$, pick a collection $\mathcal{B}(s)$ of simple circuits in $\bar{\mathcal{V}}$ such that $\mathcal{B}(s)$ generates the fundamental group of the connected component of s in $\bar{\mathcal{V}}$.

The following theorem is proven in Brink's dissertation (see [5]):

Theorem 2.1. *For $s \in S$, the centralizer $C(s)$ is generated by*

$$\{s\} \cup A \cup B,$$

where

$$A = \left\{ v u_{ts_k} v^{-1} \;\middle|\; v = v_{s_1 s} v_{s_2 s_1} \cdots v_{s_k s_{k-1}};\ t, s_i \in S;\ m_{ts_k} \textit{ even } m_{s_i s_{i-1}} \textit{ odd} \right\}$$

and

$$B = \left\{ v_{s_1 s} v_{s_2 s_1} \cdots v_{s_k s_{k-1}} v_{s s_k} \;\middle|\; \left\{ [s s_1], [s_1 s_2], \ldots, [s_k s] \right\} \in \mathcal{B}(s) \right\}.$$

Remark. Brink's characterization of $C(s)$ is made in terms of the root system corresponding to (W, S). The above theorem is a "translation" of Brink's work into the language of the generating set S.

An element of A corresponds to an "odd path" in $\mathcal{V}$, followed by an "even" reflection, and then a retracing of the odd path. An element of B corresponds to an "odd circuit" in $\mathcal{V}$. We will use this informal terminology in the sequel.

The following two lemmas (both proven in [4]) will be very useful:

Lemma 2.2. *Let $G \leqslant W$ and $|G| < \infty$. Then there exists a spherical subgroup W_T of W and an element $w \in W$ such that $G \leqslant w W_T w^{-1}$.*

Lemma 2.3. *Let $T \subseteq S$, $|T| = 3$. Then W_T is spherical if and only if the vertices of T form a triangle in $\mathcal{V}$ and the multiset of labels for this triangle lies in the set*

$$\left\{ \{2,3,3\}, \{2,3,4\}, \{2,3,5\} \right\} \cup \left\{ \{2,2,k\} \mid k = 2, 3, \ldots \right\}.$$

Finally, we have the following crucial lemma, whose proof is not difficult. If s and t are conjugate in W, we denote this relationship by $s \sim t$.

Lemma 2.4. *Let $s, t \in \mathcal{V}$, $s \neq t$. Then $s \sim t$ if and only if there is a path $\{[s s_1], [s_1 s_2], \ldots, [s_k t]\}$ from s to t in $\mathcal{V}$, all of whose edges have odd labels. In this case, $v s v^{-1} = t$, where $v = v_{s_k t} v_{s_{k-1} s_k} \cdots v_{s_1 s_2} v_{s s_1}$.*

3. Triangle subgroups of W

Suppose that $G \leqslant W$ is a triangle subgroup of W. That is, G is isomorphic to a Coxeter group with presentation

$$\left\langle a, b, c \;\middle|\; a^2, b^2, c^2, (ab)^p, (bc)^q, (ca)^r \right\rangle.$$

Let $|G| < \infty$. Because (W, S) is 2D, Lemma 2.2 implies that $G \leqslant w W_{\{s,t\}} w^{-1}$ for some $w \in W$ and some edge $[st]$. Therefore, since the number of conjugacy classes of finite subgroups of rank-2 spherical subgroups of W is clearly finite, we may assume that $|G| = \infty$. (In particular, by Lemma 2.3, at most one of $\{p, q, r\}$ is 2.)

There are three affine Euclidean triangle groups, namely $(3,3,3)$, $(2,4,4)$, and $(2,3,6)$. The following theorem of Krammer allows us to immediately eliminate from consideration such triangle subgroups.

Theorem 3.1. *Let W' be an affine Euclidean Coxeter group of rank 3, and let $W' \leqslant W$. Then W' lies inside some affine Euclidean parabolic subgroup of W of rank at least 3.*

Since the group W under consideration contains no such parabolic subgroups (by hypothesis!), there can be no affine Euclidean triangle subgroups of any sort in W. We must therefore consider only the "hyperbolic" triangle groups (p,q,r). (These are distinguished by the condition $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$.)

Our goal is to prove the following:

Proposition 3.2. *Every hyperbolic triangle subgroup of W must be conjugate to a subgroup of a rank-3 standard parabolic subgroup of W. Moreover, for any such rank-3 parabolic subgroup, there are only finitely many conjugacy classes of triangle subgroups.*

Theorem 1.1 follows from this, as there are only finitely many triangles in $\mathcal{V}$.

4. The proof of Proposition 3.2

Let G be as in the previous section. Each pair of generators from $\{a,b,c\}$ generates a finite subgroup of W, which must therefore (by Lemma 2.2) lie inside a conjugate of a rank-2 spherical subgroup of W. There are a few cases to consider.

Case 1: *Distinct pairs from $\{a,b,c\}$ correspond to distinct edges of $\mathcal{V}$.* There are generators a_1,a_2,b_1,b_2,c_1,c_2 in S, and words w_1,w_2,w_3,u_1,u_2,u_3 in W such that $[a_2b_1]$, $[b_2c_1]$, and $[c_2a_1]$ are all edges in $\mathcal{V}$ and

$$w_1aw_1^{-1}=a_2 \quad \text{and} \quad w_1bw_1^{-1}\in\{u_1a_2u_1^{-1},u_1b_1u_1^{-1}\},$$
$$w_2bw_2^{-1}=b_2 \quad \text{and} \quad w_2cw_2^{-1}\in\{u_2b_2u_2^{-1},u_2c_1u_2^{-1}\},$$

and

$$w_3cw_3^{-1}=c_2 \quad \text{and} \quad w_3aw_3^{-1}\in\{u_3c_2u_3^{-1},u_3a_1u_3^{-1}\}.$$

Remark. Note that if ab has order 2, it is conceivable that $w_1aw_1^{-1}=(a_2b_1)^{n_1/2}$. However, this quickly leads to a contradiction. In this case, a is not a reflection in (W,S). Thus $w_3aw_3^{-1}=(a_1c_2)^{n3/2}$ must hold, so that ac also has order 2, forcing $|G|<\infty$, which we assumed is not the case.

Let us first assume that $w_1bw_1^{-1}=u_1b_1u_1^{-1}$, $w_2cw_2^{-1}=u_2c_1u_2^{-1}$, and $w_3aw_3^{-1}=u_3a_1u_3^{-1}$ all hold. In this case, $a_1\sim a_2$, $b_1\sim b_2$, and $c_1\sim c_2$. Let v_1, v_2, and v_3 be respective conjugating elements for these pairs, chosen as in Lemma 2.4. Moreover, it is not hard to see that the words w_i and u_i can be chosen so that $u_1 = (b_1a_2)^{k_1}$, $u_2 = (c_1b_2)^{k_2}$, and $u_3 = (a_1c_2)^{k_3}$, for some $k_i < n_i$, where the labels on the edges $[a_2b_1]$, $[b_2c_1]$, and $[c_2a_1]$ are n_1, n_2, and n_3, respectively. (More precisely, $k_i \leqslant [\frac{n_i}{4}]$ if n_i is even, and $k_i \leqslant \frac{n_i-3}{2}$ if n_i is odd. Similar computations are performed in each of [1–3].)

We can then compute:

$$a_2 = w_1 a w_1^{-1} = w_1 w_3^{-1} u_3 v_1^{-1} a_2 v_1 u_3^{-1} w_3 w_1^{-1}.$$

From this we get $w_1 w_3^{-1} = \bar{a} v_1 u_3^{-1}$, for some $\bar{a} \in C(a_2)$ whose form is dictated by Theorem 2.1. We denote by x_1 the right-hand side of this equation. Similarly, $w_2 w_1^{-1} = \bar{b} v_2 u_1^{-1} =: x_2$, and $w_3 w_2^{-1} = \bar{c} v_3 u_2^{-1} =: x_3$, for some $\bar{b} \in C(b_2)$ and $\bar{c} \in C(c_2)$.

Some obvious cancellation may occur between the words $\bar{a}$ and v_1, if $\bar{a}$ ends with a product of "odd" terms v_{st} with which v_1 begins. We perform this cancellation, and any similar cancellation that arises in $\bar{b}v_2$ and $\bar{c}v_3$. Finally, there could be a small amount of cancellation in multiplying the words u_i^{-1} with the respective products given above. After this last cancellation is performed, the result is a trio of geodesic words representing the same group elements as x_i. (Here we appeal to the solution to the Word Problem for Coxeter groups given by Tits in [10].)

Notice that

$$x_1 x_3 x_2 = w_1 w_3^{-1} \cdot w_3 w_2^{-1} \cdot w_2 w_1^{-1} = 1.$$

We claim (as in [1] and subsequent work) that this equality forces each of the words x_i to be short, and G to be conjugate to the parabolic subgroup $W_{\{a_1,b_1,c_1\}}$. The proof of this claim uses the same arguments as [1]. We indicate the thrust of this proof by considering a few cases, leaving the remaining cases to the reader.

First, assume that each of the words u_i is trivial. Furthermore, we assume first that $x_2 = 1$. After canceling common terms v_{st} in x_1 and x_3, x_1 must end either with an odd term $v_{a_1\alpha}$ or an even term $u_{\alpha a_1}$ for some generator α, and x_3 must begin either with an odd term $v_{\gamma c_2}$ or an even term $u_{\gamma c_2}$, for some generator γ.

It is easy to see that if one term is odd and the other even, then very little cancellation can occur, and the product x_1x_3 cannot be trivial, as required. Therefore either both are even or both are odd.

Suppose that both are even. Cancellation of more than a pair of letters results only if $\alpha = c_2$ and $\gamma = a_1$, and in this case $u_{\alpha a_1} u_{\gamma c_2} = a_1 c_2$. Further cancellation (between additional terms of x_1 and x_3) can occur only if a_1 precedes $u_{c_2 a_1}$ in x_1 and c_2 follows $u_{a_1 c_2}$ in x_3. Furthermore, Theorem 2.1 shows that there can be no further cancellation in any case. It follows that $a_1 = a_2$ and $c_2 = c_1$, and $b_1 = b_2$ follows from $x_2 = 1$ in the first place. Moreover, $x_1 = x_3 = a_1 u_{c_1 a_1}$ implies (after short computation) that $w_1 = w_3$; $w_1 = w_2$ follows from $x_2 = 1$. We at last have $G = w_1 W_{\{a_1,b_1,c_1\}} w_1^{-1}$.

Similar arguments hold if both terms in the penultimate paragraph are odd. In this case we have $v_{a_1\alpha} v_{\gamma c_2}$, which admits cancellation only if $\alpha = c_2$ and $\gamma = a_1$, yielding the product $c_2 a_1$. This product admits no further cancellation, regardless of the form of the remaining portions of x_1 and x_3. If, however, $a_2 = c_2$ and $a_1 = c_1$ both hold, x_1 ends with $c_1 v_{a_1 c_1}$, and x_3 begins with $v_{a_1 c_1} a_1$, then in the product $x_1 x_3$ the entirety of these terms cancel, and as no further cancellation can occur, we conclude that $x_1 = x_3 = a_1 v_{c_1 a_1}$. As before, $b_1 = b_2$ and $w_1 = w_2$ follow from $x_2 = 1$, and from the above computations we obtain $w_1 = c_1 v_{a_1 c_1} w_3$, which does not effect the fact that $G = w_1 W_{\{a_1,b_1,c_2\}} w_1^{-1}$ still obtains (we have merely performed a "twist" about the edge $[a_1c_2]$).

If any one of the words x_i is trivial (at most one can be), we can relabel to guarantee that $x_2 = 1$. Moreover, similar arguments hold even if x_2 is not trivial. Finally, we note that $u_i = 1$ must hold for $i = 1, 2, 3$. This can be shown by methods similar to those used above. (Roughly speaking, since the words u_i are "short," if they were to be nontrivial, their inclusion in the product $x_1x_3x_2$ would obstruct cancellation between terms of the form u_{st} and v_{st}, forcing $x_1x_3x_2$ to be nontrivial.)

Now let us suppose that $w_1 b w_1^{-1} = u_1 a_2 u_1^{-1}$ holds (while $w_2 c w_2^{-1} = u_2 c_1 u_2^{-1}$ and $w_3 a w_3^{-1} = u_3 a_1 u_3^{-1}$). We claim that $b_1 a_2$ must have odd order and that $u_1 = v_{a_2 b_1}$, so that $w_1 b w_1^{-1} = b_1$, so we really have nothing new.

Were $b_1 a_2$ to have even order, we note that u_1 could be written $(a_2 b_1)^k$, where $1 \leqslant k \leqslant [\frac{n_1}{4}] + 1$. Arguing as in the cases above, we obtain x_1 and x_3 as before, whereas now $x_2 = \bar{b} v_2 u_1^{-1}$ for v_2 conjugating a_2 to b_2.

Considering first the case $x_3 = 1$, we have

$$1 = x_1 x_2 = \bar{a} v_1 u_3^{-1} \bar{b} v_2 u_1^{-1}.$$

However, because u_1 is "short," it obstructs complete cancellation in the above product, unless $u_1 = 1$. This is a contradiction, as in this $a = b$ results. Even when $x_3 \neq 1$, $u_1 = 1$ must hold in order for $x_1x_3x_2$ to be trivial.

If instead $b_1 a_2$ has odd order, we have $w_1 b w_1^{-1} = u_1 a_2 u_1^{-1} = u_1 v_{b_1 a_2} b_1 v_{a_2 b_1} u_1^{-1}$, and the word $u_1 v_{b_1 a_2}$ can be written $(a_2 b_1)^k$ for some $0 \leqslant k \leqslant \frac{n_1}{2} - 1$. We may now argue as before, replacing the word u_1 with $u_1 v_{b_1 a_2}$; complete cancellation will be possible only when this word is trivial, forcing $u_1 = v_{a_2 b_1}$, forcing $w_1 b w_1^{-1} = b_1$, as desired.

Similarly, both $w_2 c w_2^{-1} = u_2 c_1 u_2^{-1}$ and $w_3 a w_3^{-1} = u_3 a_1 u_3^{-1}$ must hold, and we may argue as we did initially.

To summarize, we have shown that in case each 2-generated "parabolic" subgroup of $G = \langle\{a, b, c\}\rangle$ corresponds to a different edge of $\mathcal{V}$, $G = w W_{\{a_1,b_1,c_1\}} w^{-1}$ holds for some $a_1, b_1, c_1 \in S$, so G is a rank-3 parabolic subgroup of W, of which there are finitely many. The first statement of Proposition 3.2 is thus shown; the second will follow immediately once we have considered Cases 2 and 3 below.

Case 2: *All three pairs from $\{a, b, c\}$ correspond to the same edge of $\mathcal{V}$.* Now let $x, y \in \mathcal{V}$, $w_1, w_2, w_3 \in W$, and $u_1, u_2, u_3 \in W_{\{x,y\}}$ such that

$$w_1 a w_1^{-1} = x \quad \text{and} \quad w_1 b w_1^{-1} \in \{u_1 x u_1^{-1}, u_1 y u_1^{-1}\},$$
$$w_2 b w_2^{-1} = x \quad \text{and} \quad w_2 c w_2^{-1} \in \{u_2 x u_2^{-1}, u_2 y u_2^{-1}\},$$

and

$$w_3 c w_3^{-1} = x \quad \text{and} \quad w_3 a w_3^{-1} \in \{u_3 x u_3^{-1}, u_3 y u_3^{-1}\}.$$

Remark. Note that each of p, q, and r must divide the label on $[xy]$.

We consider only the case in which $w_1 b w_1^{-1} = u_1 y u_1^{-1}$, $w_2 c w_2^{-1} = u_2 y u_2^{-1}$, and $w_3 a w_3^{-1} = u_3 y u_3^{-1}$. (As in Case 1, the other possibilities are analogous.) Arguing as in Case 1, we obtain $w_1 w_3^{-1} = \bar{x}_1 v^{-1} u_3^{-1}$, $w_2 w_1^{-1} = \bar{x}_2 v^{-1} u_1^{-1}$, and $w_3 w_2^{-1} = \bar{x}_3 v^{-1} u_2^{-1}$, where $\bar{x}_i \in C(x)$ for each $i = 1, 2, 3$ and $vxv^{-1} = y$. Thus

$$\bar{x}_1 v^{-1} u_3^{-1} \bar{x}_3 v^{-1} u_2^{-1} \bar{x}_2 v^{-1} u_1^{-1} = 1. \tag{5}$$

The case in which the label $m = m_{xy}$ of $[xy]$ is odd is easier, as in that case we can take $v = (xy)^{(m-1)/2}$, and we obtain

$$v^{-1} u_i^{-1} \in \left\{ (yx)^{(m-1)/2}, (yx)^{(m-3)/2} y, (yx)^{(m-3)/2}, \ldots, yx, y \right\}.$$

It is now not hard to show that if any one of the words $\bar{x}_i$ contains a generator of $C(x)$ corresponding to an odd path or circuit of length greater than 2, then the product in (5) cannot be trivial. Moreover, one can show that $\mathcal{V}$ must contain a triangle with generators $\{x, y, z\}$ and edge labels $[xy] = m$, $[xz] = 2$, and $[yx] \in \{3, 4\}$, where all three of p, q, and r divide m.

We note that because none of the words $\bar{x}_i$ can be very long, there are only finitely many valid choices for w_2 and w_3 given a fixed choice of w_1. Thus the number of conjugacy classes of triangle subgroups corresponding to a given triangle $\{x, y, z\}$ as above is finite.

The case in which the label of $[xy]$ is even is analogous, and yields no new possibilities.

Remark. Case 2 can in fact occur: the elements x, y, and $zyxyxyxyz$ generate a triangle subgroup of the form $(7, 7, 7)$ inside of the parabolic triangle subgroup $(2, 3, 7)$ generated by $\{x, y, z\}$. (That is, $m_{xy} = 7$, $m_{xz} = 2$, and $m_{yz} = 3$.) Here, up to relabeling, the product in (5) has the form

$$1 = (yx)^3 (zy) x (yz)(xy)^3 \cdot (yx)^3 \cdot 1 \cdot z \cdot (yx)^3 \cdot xyx \cdot zx \cdot (yx)^3 \cdot x.$$

(The $\cdot$'s separate the nine terms of (5) from one another.)

The group $(2, 4, 7)$ contains a copy of $(7, 7, 7)$ in much the same way, as the reader is invited to check.

Case 3: *Two pairs from $\{a, b, c\}$ correspond to a single edge of $\mathcal{V}$.* This case is entirely analogous to the last two, with only slightly different computations. There are a number of subcases to consider, but we obtain no new possibilities.

In each of the three cases, the triangle subgroup $G = \langle \{a, b, c\} \rangle$ must be conjugate to a subgroup of a hyperbolic standard parabolic subgroup of (W, S). Since the arguments all demonstrate that any rank-3 standard parabolic subgroup of (W, S) can contain only finitely many conjugacy classes of such triangle subgroups, the second statement of Proposition 3.2 follows as promised. Theorem 1.1 follows at once from this.

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